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SEIBERG-WITTEN THEORY AND CALOGERO-MOSER SYSTEMS *

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Abstract

We present a brief account of a series of recent results on twisted and untwisted elliptic Calogero-Moser systems, and on their fundamental role in the Seiberg-Witten solution of gauge theories with one massive hypermultiplet in the adjoint representation of an arbitrary gauge algebra $\mathcal{G}$.

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I. INTRODUCTION

Over the past few years, a tremendous amount of progress has been made in the understanding of strongly coupled supersymmetric gauge and string theories. These advances were driven in large part by the Seiberg-Witten solution of $\mathcal{N}=2$ supersymmetric Yang-Mills theory for $SU(2)$ gauge group [1].

Of central interest to many of these developments is the 4-dimensional supersymmetric Yang-Mills theory with maximal supersymmetry, $\mathcal{N}=4$, and with arbitrary gauge algebra $\mathcal{G}$. In the present paper, we shall consider a generalization of this theory, in which a mass term is added for part of the $\mathcal{N}=4$ gauge multiplet, softly breaking the $\mathcal{N}=4$ symmetry to $\mathcal{N}=2$. As an $\mathcal{N}=2$ supersymmetric theory, the theory has a $\mathcal{G}$-gauge multiplet, and a hypermultiplet in the adjoint representation of $\mathcal{G}$ with mass m. This generalized theory shares many of the properties of the $\mathcal{N}=4$ theory : it has the same field content; it is ultra-violet finite; it has vanishing renormalization group β-function, and it is expected to have Montonen-Olive duality symmetry. For vanishing hypermultiplet mass $m=0$, the $\mathcal{N}=4$ theory is recovered. For $m\to\infty$, the limiting theory is one of many interesting $\mathcal{N}=2$ supersymmetric Yang-Mills theories. The possibilities for, say, $\mathcal{G}=SU(N)$, include theories with any number of hypermultiplets in the fundamental representation of $SU(N)$, or with product gauge algebras $SU(N_1)\times SU(N_2)\times\cdots\times SU(N_p)$, and hypermultiplets in the fundamental and bi-fundamental representations of these product algebras.

Remarkably, the Seiberg-Witten theory for $\mathcal{N}=2$ supersymmetric Yang-Mills theory for arbitrary gauge algebra $\mathcal{G}$ appears to be intimately related with the existence of certain classical mechanics integrable systems. This relation was first suspected on the basis of the similarity between the Seiberg-Witten curves and the spectral curves of certain integrable models [2]. Then, general arguments showed that Seiberg-Witten Ansatz naturally produces integrable structures [3]. For the $\mathcal{N}=2$ supersymmetric Yang-Mills theory with massive hypermultiplet, the relevant integrable system appears to be the *elliptic Calogero-Moser system.* For $SU(N)$ gauge group, Donagi and Witten [3] proposed that the spectral curves of the $SU(N)$ Hitchin system should play the role of the Seiberg-Witten curves. Krichever (in unpublished work), Gorsky and Nekrasov, and Martinec [4] recognized that the $SU(N)$ Hitchin system spectral curves are identical to those of the $SU(N)$ elliptic Calogero-Moser integrable system.

That the $SU(N)$ elliptic Calogero-Moser curves (and associated Seiberg-Witten differential) do indeed provide the Seiberg-Witten solution for the $\mathcal{N}=2$ theory with one massive hypermultiplet was fully established by the authors in [5]. There, it was shown that the effective prepotential $\mathcal{F}$ reproduces the logarithmic singularities predicted by perturbation theory; that $\mathcal{F}$ satisfies a renormalization group type equation which determines instanton contributions to any order; and that the prepotential in the limit of large hypermultiplet mass m reproduces the prepotentials for $\mathcal{N}=2$ super Yang-Mills theory with any number of hypermultiplets in the fundamental representation of the gauge group.

The fundamental problem in Seiberg-Witten theory is to determine the Seiberg-Witten curves and differentials, corresponding to an $\mathcal{N}=2$ supersymmetric gauge theory with arbitrary gauge algebra $\mathcal{G}$, and a massive hypermultiplet in an arbitrary representation R of $\mathcal{G}$, subject to the constraint of asymptotic freedom or conformal invariance. With the correspondence between Seiberg-Witten curves and the spectral curves of classical mechanics integrable systems [3], this problem is equivalent to determining a general integrable system, associated with the Lie algebra $\mathcal{G}$ and the representation R.

The $\mathcal{N}=2$ theory for arbitrary gauge algebra $\mathcal{G}$ and with one massive hypermultiplet in the adjoint representation was one such outstanding case when $\mathcal{G}\neq SU(N)$. Actually, as discussed previously, upon taking suitable limits, this theory contains a very large number of models with smaller hypermultiplet representations R, and in this sense has a universal aspect. It appeared difficult to generalize directly the Donagi-Witten construction of Hitchin systems to arbitrary $\mathcal{G}$, and it was thus natural to seek this generalization directly amongst the elliptic Calogero-Moser integrable systems. It has been known now for a long time, thanks to the work of Olshanetsky and Perelomov [6], that Calogero-Moser systems can be defined for any simple Lie algebra. Olshanetsky and Perelomov also showed that the Calogero-Moser systems for *classical* Lie algebras were integrable, although the existence of a spectral curve (or Lax pair with spectral parameter) as well as the case of exceptional Lie algebras remained open.

The purpose of this paper is to review the resolution of the above problems by the following results which were obtained in [7], [8], [9].

• The elliptic Calogero-Moser systems defined by an arbitrary simple Lie algebra $\mathcal{G}$ admit Lax pairs with spectral parameters.

• The correspondence between elliptic $\mathcal{G}$ Calogero-Moser systems and $\mathcal{N}=2$ supersymmetric $\mathcal{G}$ gauge theories with matter in the adjoint representation holds directly when the Lie algebra $\mathcal{G}$ is simply-laced. When $\mathcal{G}$ is not simply-laced, the correspondence is with new integrable models, *the twisted elliptic Calogero-Moser systems* introduced in [7,8].

• Twisted elliptic Calogero-Moser systems admit a Lax pair with spectral parameter [7].

• In the scaling limit $m=Mq^{-\frac{1}{2}\delta}\to\infty$, M fixed, the twisted (respectively untwisted) elliptic $\mathcal{G}$ Calogero-Moser systems tend to the Toda system for $(\mathcal{G}^{(1)})^\vee$ (respectively $\mathcal{G}^{(1)}$) for $\delta=\frac{1}{h_\mathcal{G}^\vee}$ (respectively $\delta=\frac{1}{h_\mathcal{G}}$). Here $h_\mathcal{G}$ and $h_\mathcal{G}^\vee$ are the Coxeter and the dual Coxeter numbers of $\mathcal{G}$ [8].

The remainder of this paper is organized as follows. In §II, we review the set-up and basic constructions of Seiberg-Witten theory. In §III, we discuss the elliptic Calogero-Moser systems, and present the new twisted elliptic Calogero-Moser systems introduced in [7,8]. In §IV, we show how these systems tend to Toda systems in certain limits. In §V, we discuss their integrability properties and present their Lax pairs with spectral parameter. Finally, in §VI, we discuss the Seiberg-Witten solution for the $\mathcal{N}=2$ supersymmetric Yang-Mills theories and a massive hypermultiplet in the adjoint representation of an arbitrary gauge algebra $\mathcal{G}$. A prior review of these results has appeared in [10], where further results on spin Calogero-Moser systems are also presented.

II. SEIBERG-WITTEN THEORY

The starting point for Seiberg-Witten theory is an $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge algebra $\mathcal{G}$ and hypermultiplets in a representation R of $\mathcal{G}$ with masses m_j. The microscopic Lagrangian is completely fixed by $\mathcal{N}=2$ supersymmetry in terms of the gauge coupling g and the instanton angle θ, and is given by

$$\mathcal{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a} + D_\mu \bar{\phi} D^\mu \phi + \mathrm{tr}[\bar{\phi}, \phi]^2 + \cdots \tag{2.1}$$

where we have neglected hypermultiplet and fermion terms.

The low energy effective theory corresponding to this model can be analyzed by studying first the structure of the vacuum. $\mathcal{N}=2$ supersymmetric vacuum states can occur whenever the vacuum energy is exactly zero, which is achieved for constant scalar fields ϕ for which the potential energy term vanishes. This requires $[\bar{\phi}, \phi] = 0$, so that the vacuum expectation value of ϕ is a linear combination of the Cartan generators h_j of the gauge algebra $\mathcal{G}$,

$$< \phi > = \sum_{j=1}^{n} a_j h_j \qquad\qquad n = \mathrm{rank}\ \mathcal{G} \tag{2.2}$$

Here, the complex parameters a_j are usually referred to as the quantum moduli, or also as the quantum order parameters of the $\mathcal{N}=2$ vacua.

For generic values of the parameters a_j, the $\mathcal{G}$-gauge symmetry will be broken down to $U(1)^n/\mathrm{Weyl}(\mathcal{G})$, and the low energy theory is that of n different Coulomb fields, up to global identifications by $\mathrm{Weyl}(\mathcal{G})$. The low energy effective Lagrangian is invariant under $\mathcal{N}=2$ supersymmetry and thus given by

$$\mathcal{L}_{\mathrm{effective}} = \frac{1}{4}\mathrm{Im}(\tau_{ij}) F^i_{\mu\nu} F^{\mu\nu j} + \frac{1}{4}\mathrm{Re}(\tau_{ij}) F^i_{\mu\nu} \tilde{F}^{\mu\nu j} + \partial_\mu \bar{\phi}^j \partial^\mu \phi_{Dj} + \mathrm{fermions} \tag{2.3}$$

Here, the dual gauge scalar ϕ_D and the gauge coupling function τ_{ij} are both given in terms of the prepotential $\mathcal{F}$

$$\phi_{Dj} = \frac{\partial \mathcal{F}(\phi)}{\partial \phi_j} \qquad \tau_{ij} = \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi_i \partial \phi_j} \tag{2.4}$$

The form of the effective Lagrangian (2.3) is the same for any of the values of the complex moduli of $\mathcal{N}=2$ vacua, with the understanding that the fields ϕ_j take on the expectation value $<\phi_j>=a_j$. Since the prepotential $\mathcal{F}(\phi)$ is a function of the fields ϕ only, but not of derivatives of ϕ, the prepotential will be completely determined by its values on the vacuum expectation values of the field, namely by its values on the quantum order parameters a_j.

The object of Seiberg-Witten theory is the determination of the prepotential $\mathcal{F}(a_j)$, from which the entire low energy effective action will be known. This is achieved by exploiting the physical conditions satisfied by $\mathcal{F}$ [1],

(1) $\mathcal{F}(a_j)$ is complex analytic in a_j in view of $\mathcal{N}=2$ supersymmetry.
(2) The matrix $\mathrm{Im}\ \tau_{ij}=\mathrm{Im}\,\partial_i\partial_j\mathcal{F}$ is positive definite, since by (2.3), it coincides with the metric on the kinetic terms for the gauge fields A_j.
(3) The large a_j behavior is known from perturbative quantum field theory calculations and asymptotic freedom, and is given by $\mathcal{F}(a)\sim\sum_\alpha(\alpha\cdot a)^2\ln(\alpha\cdot a)^2-\sum_w(w\cdot a+m)^2\ln(w\cdot a+m)^2$, where α and w are respectively the roots of $\mathcal{G}$ and the weights of the representation R.

As a result of (1) and (2), $\mathcal{F}$ cannot be a single-valued function of the a_j. For if it were, $\mathrm{Im}\ \tau_{ij}$ would be both harmonic and bounded from below, implying that it must be independent of a_j. But from (3) we know that τ_{ij} is neither constant nor single valued. The multiple-value ambiguity does not affect the physics of the low energy effective action since it may be related to electric-magnetic duality, as shown by Seiberg and Witten [1].

A natural setting in which the above monodromy problem may be solved is provided by families of Riemann surfaces, called the Seiberg-Witten spectral curves, denoted by Γ. Indeed, letting the quantum moduli a_j correspond to moduli of the Riemann surfaces, there is automatically a complex analytic period matrix, whose imaginary part is positive definite, and whose monodromy group corresponds to the modular group of the surface. The general set-up of the Seiberg-Witten solution, expected for arbitrary gauge algebra $\mathcal{G}$ with rank n and general hypermultiplet representation is as follows.

(1) The Seiberg-Witten curve is a family of Riemann surfaces $\Gamma(u_1,\cdots,u_n)$ dependent on n auxiliary complex parameters u_j, which are related to the quantum moduli a_j. The Seiberg-Witten curve will also depend upon the gauge coupling g and θ-angle and on the hypermultiplet masses m_k.
(2) The Seiberg-Witten differential 1-form $d\lambda$ is meromorphic on Γ, with residues which are linear in the hypermultiplet masses m_k. Since the hypermultiplet masses receive no quantum corrections as a_j varies, the derivatives $\partial(d\lambda)/\partial a_j$ are holomorphic 1-forms.

(3) The quantum moduli and the prepotential are given by

$$a_j = \frac{1}{2\pi i}\oint_{A_j} d\lambda \qquad\qquad a_{Dj} = \frac{\partial \mathcal{F}}{\partial a_j} = \frac{1}{2\pi i}\oint_{B_j} d\lambda \tag{2.5}$$

for a suitable set of cycles A_j and B_j on Γ.

Shortly after the initial work of Seiberg and Witten, curves and differentials were proposed for a general classical gauge group, with and without hypermultiplets in the fundamental representation. Use was made of the R-charge assignments of the fields, the singularity structure of the degenerations of the Seiberg-Witten curve and much educated guess work (see e.g. [11] for reviews). The monodromies and instanton corrections in the corresponding prepotential were determined in [31]. More recently, Seiberg-Witten curves for many other theories have been found, based on integrable models [12-14], M-Theory [15], and geometric engineering [16]. Monodromies and instanton corrections have been determined in several important cases, including for the $SU(N)$ gauge theory, with hypermultiplet in the symmetric or anti-symmetric representation [17]. The prepotential has also been analyzed using methods from Whitham theory [18, 30] and WDVV equations [19].

III. TWISTED AND UNTWISTED CALOGERO-MOSER SYSTEMS

a) The $SU(N)$ Elliptic Calogero-Moser System

The original elliptic Calogero-Moser system is the system defined by the Hamiltonian

$$H(x,p) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j} \wp(x_i - x_j) \tag{3.1}$$

Here m is a mass parameter, and $\wp(z)$ is the Weierstrass $\wp$-function, defined on a torus $\mathbf{C}/(2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z})$. As usual, we denote by $\tau = \omega_2/\omega_1$ the moduli of the torus, and set $q = e^{2\pi i\tau}$. The well-known trigonometric and rational limits with respective potentials

$$\begin{aligned} H_{\text{trig}} &= \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j} \frac{1}{4\,\text{sh}^2\,(\frac{x_i - x_j}{2})} \\ H_{\text{rat}} &= \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j} \frac{1}{(x_i - x_j)^2} \end{aligned}$$

arise in the limits $\omega_1 = -i\pi, \omega_2 \to \infty$ and $\omega_1, \omega_2 \to \infty$. All these systems have been shown to be completely integrable in the sense of Liouville, i.e. they all admit a complete set of integrals of motion which are in involution [20-22]. For a recent review of some applications of these models see [23].

Our considerations require however a notion of integrability which is in some sense more stringent, namely the existence of a Lax pair $L(z)$, $M(z)$ with spectral parameter z. Such a Lax pair was obtained by Krichever [24] in 1980. He showed that the Hamiltonian system (3.1) is equivalent to the Lax equation $\dot{L}(z) = [L(z), M(z)]$, with $L(z)$ and $M(z)$ given by the following $N \times N$ matrices

$$\begin{aligned} L_{ij}(z) =& p_i \delta_{ij} - m(1-\delta_{ij}) \Phi(x_i - x_j, z) \\ M_{ij}(z) =& m\delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - m(1-\delta_{ij}) \Phi'(x_i - x_j, z). \end{aligned} \tag{3.2}$$

The function $\Phi(x, z)$ is defined by

$$\Phi(x, z) = \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{x\zeta(z)}, \tag{3.3}$$

where $\sigma(z)$, $\zeta(z)$ are the usual Weierstrass σ and ζ functions on the torus $\mathbf{C}/(2\omega_1 \mathbf{Z} + 2\omega_2 \mathbf{Z})$. The function $\Phi(x, z)$ satisfies the key functional equation

$$\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = (\wp(x) - \wp(y))\Phi(x + y, z). \tag{3.4}$$

It is well-known that functional equations of this form are required for the Hamilton equations of motion to be equivalent to the Lax equation $\dot{L}(z) = [L(z), M(z)]$ with a Lax pair of the form (3.2). It is a relatively recent result of Braden and Buchstaber [25] that, conversely, general functional equations of the form (3.4) essentially determine $\Phi(x, z)$.

b) Calogero-Moser Systems defined by Lie Algebras

Olshanetsky and Perelomov [6] showed that the Hamiltonian system (3.1) is only one of a series associated with each simple Lie algebra. Given any simple Lie algebra $\mathcal{G}$, Olshanetsky and Perelomov [6] introduced the system with Hamiltonian

$$H(x, p) = \frac{1}{2} \sum_{i=1}^{r} p_i^2 - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \wp(\alpha \cdot x), \tag{3.5}$$

where r is the rank of $\mathcal{G}$, $\mathcal{R}(\mathcal{G})$ denotes the set of roots of $\mathcal{G}$, and the $m_{|\alpha|}$ are mass parameters. To preserve the invariance of (3.5) under the Weyl group, the parameters $m_{|\alpha|}$ depend only on the orbit $|\alpha|$ of the root α, and not on the root α itself. In the case of $A_{N-1} = SU(N)$, it is common practice to use N pairs of dynamical variables (x_i, p_i), since the roots of A_{N-1} lie conveniently on a hyperplane in $\mathbf{C}^N$. The dynamics of the system are unaffected if we shift all x_i by a constant, and the number of degrees of freedom is effectively $N - 1 = r$. Now the roots of $SU(N)$ are given by $\alpha = e_i - e_j$, $1 \leq i, j \leq N$, $i \neq j$. Thus we recognize the original elliptic Calogero-Moser system as the special case of

(3.5) corresponding to A_{N-1}. Olshanetsky and Perelomov constructed a Lax pair for all these systems with *classical* Lie algebras, without spectral parameter [6].

c) Twisted Calogero-Moser Systems defined by Lie Algebras

It turns out that the Hamiltonian systems (3.5) are not the only natural extensions of the basic elliptic Calogero-Moser system. A subtlety arises for simple Lie algebras $\mathcal{G}$ which are not simply-laced, i.e., algebras which admit roots of uneven length. This is the case for the algebras B_n, C_n, G_2, and F_4 in Cartan's classification. For these algebras, the following *twisted* elliptic Calogero-Moser systems were introduced by the authors in [7,8]

$$H^{\mathrm{twisted}} = \frac{1}{2}\sum_{i=1}^{r} p_i^2 - \frac{1}{2}\sum_{\alpha\in\mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \wp_{\nu(\alpha)}(\alpha\cdot x). \tag{3.6}$$

Here the function $\nu(\alpha)$ depends only on the length of the root α. If $\mathcal{G}$ is simply-laced, we set $\nu(\alpha) = 1$ identically. Otherwise, for $\mathcal{G}$ non simply-laced, we set $\nu(\alpha) = 1$ when α is a long root, $\nu(\alpha) = 2$ when α is a short root and $\mathcal{G}$ is one of the algebras B_n, C_n, or F_4, and $\nu(\alpha) = 3$ when α is a short root and $\mathcal{G} = G_2$. The *twisted* Weierstrass function $\wp_\nu(z)$ is defined by

$$\wp_\nu(z) = \sum_{\sigma=0}^{\nu-1} \wp(z + 2\omega_a \frac{\sigma}{\nu}), \tag{3.7}$$

where ω_a is any of the half-periods ω_1, ω_2, or $\omega_1+\omega_2$. Thus the twisted and untwisted Calogero-Moser systems coincide for $\mathcal{G}$ simply laced. The original motivation for twisted Calogero-Moser systems was based on their scaling limits (which will be discussed in the next section) [7,8]. Another motivation based on the symmetries of Dynkin diagrams was proposed subsequently by Bordner, Sasaki, and Takasaki [26].

IV. SCALING LIMITS OF CALOGERO-MOSER SYSTEMS

a) Results of Inozemtsev for A_{N-1}

For the standard elliptic Calogero-Moser systems corresponding to A_{N-1}, Inozemtsev [27] has shown in the 1980's that in the scaling limit

$$\begin{aligned} m &= Mq^{-\frac{1}{2N}}, && q \to 0 \\ x_i &= X_i - 2\omega_2 \frac{i}{N}, && 1 \leq i \leq N \end{aligned} \tag{4.1}$$

where M is kept fixed, the elliptic A_{N-1} Calogero-Moser Hamiltonian tends to the following Hamiltonian

$$H_{\mathrm{Toda}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\Big(\sum_{i=1}^{N-1} e^{X_{i+1}-X_i} + e^{X_1-X_N}\Big) \tag{4.2}$$

The roots $e_i - e_{i+1}$, $1 \le i \le N-1$, and $e_N - e_1$ can be recognized as the simple roots of the affine algebra $A^{(1)}_{N-1}$. (For basic facts on affine algebras, we refer to [28]). Thus (4.2) can be recognized as the Hamiltonian of the Toda system defined by $A^{(1)}_{N-1}$.

b) Scaling Limits based on the Coxeter Number

The key feature of the above scaling limit is the collapse of the sum over the entire root lattice of A_{N-1} in the Calogero-Moser Hamiltonian to the sum over only simple roots in the Toda Hamiltonian for the Kac-Moody algebra $A^{(1)}_{N-1}$. Our task is to extend this mechanism to general Lie algebras. For this, we consider the following generalization of the preceding scaling limit

$$\begin{aligned} m &= M q^{-\frac{1}{2}\delta}, \\ x &= X - 2\omega_2 \delta \rho^\vee, \end{aligned} \tag{4.3}$$

Here $x = (x_i)$, $X = (X_i)$ and $\rho^\vee$ are r-dimensional vectors. The vector x is the dynamical variable of the Calogero-Moser system. The parameters δ and $\rho^\vee$ depend on the algebra $\mathcal{G}$ and are yet to be chosen. As for M and X, they have the same interpretation as earlier, namely as respectively the mass parameter and the dynamical variables of the limiting system. Setting $\omega_1 = -i\pi$, the contribution of each root α to the Calogero-Moser potential can be expressed as

$$m^2 \wp(\alpha \cdot x) = \frac{1}{2} M^2 \sum_{n=-\infty}^{\infty} \frac{e^{2\delta\omega_2}}{\mathrm{ch}(\alpha \cdot x - 2n\omega_2) - 1} \tag{4.4}$$

It suffices to consider positive roots α. We shall also assume that $0 \le \delta\, \alpha \cdot \rho^\vee \le 1$. The contributions of the $n = 0$ and $n = -1$ summands in (4.4) are proportional to $e^{2\omega_2(\delta - \delta\, \alpha \cdot \rho^\vee)}$ and $e^{2\omega_2(\delta - 1 + \delta\, \alpha\cdot\rho^\vee)}$ respectively. Thus the existence of a finite scaling limit requires that $\delta \le \delta\, \alpha \cdot \rho^\vee \le 1 - \delta$. Let α_i, $1 \le i \le r$ be a basis of simple roots for $\mathcal{G}$. If we want all simple roots α_i to survive in the limit, we must require that $\alpha_i \cdot \rho^\vee = 1,\ 1 \le i \le r$. This condition characterizes the vector $\rho^\vee$ as the *level vector*. Next, the second condition in (3.7) can be rewritten as $\delta\{1 + max_\alpha\, (\alpha \cdot \rho^\vee)\} \le 1$. But

$$h_{\mathcal{G}} = 1 + max_\alpha\, (\alpha \cdot \rho^\vee) \tag{4.5}$$

is precisely the Coxeter number of $\mathcal{G}$, and we must have $\delta \le \frac{1}{h_{\mathcal{G}}}$. Thus when $\delta < \frac{1}{h_{\mathcal{G}}}$, the contributions of all the roots except for the simple roots of $\mathcal{G}$ tend to 0. On the other hand, when $\delta = \frac{1}{h_{\mathcal{G}}}$, the highest root α_0 realizing the maximum over α in (4.5) survives. Since $-\alpha_0$ is the additional simple root for the affine Lie algebra $\mathcal{G}^{(1)}$, we arrive in this way at the following theorem, which was proved in [8]

Theorem 1. *Under the limit (4.4-4.5), with $\delta = \frac{1}{h_{\mathcal{G}}}$, and $\rho^\vee$ given by the level vector, the Hamiltonian of the elliptic Calogero-Moser system for the simple Lie algebra $\mathcal{G}$ tends to the Hamiltonian of the Toda system for the affine Lie algebra $\mathcal{G}^{(1)}$.*

(c) Scaling Limit based on the Dual Coxeter Number

If the Seiberg-Witten spectral curve of the $\mathcal{N}=2$ supersymmetric gauge theory with a hypermultiplet in the adjoint representation is to be realized as the spectral curve for a Calogero-Moser system, the parameter m in the Calogero-Moser system should correspond to the mass of the hypermultiplet. In the gauge theory, the dependence of the coupling constant on the mass m is given by

$$\tau = \frac{i}{2\pi} h_{\mathcal{G}}^{\vee} \ln \frac{m^2}{M^2} \qquad \Longleftrightarrow \qquad m = M q^{-\frac{1}{2h_{\mathcal{G}}^{\vee}}} \tag{4.6}$$

where $h_{\mathcal{G}}^{\vee}$ is the quadratic Casimir of the Lie algebra $\mathcal{G}$. This shows that the correct physical limit, expressing the decoupling of the hypermultiplet as it becomes infinitely massive, is given by (4.3), but with $\delta = \frac{1}{h_{\mathcal{G}}^{\vee}}$. To establish a closer parallel with our preceding discussion, we recall that the quadratic Casimir $h_{\mathcal{G}}^{\vee}$ coincides with the *dual Coxeter number* of $\mathcal{G}$, defined by

$$h_{\mathcal{G}}^{\vee} = 1 + max_{\alpha}\,(\alpha^{\vee}\cdot\rho), \tag{4.7}$$

where $\alpha^{\vee} = \frac{2\alpha}{\alpha^2}$ is the coroot associated to α, and $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector.

For simply-laced Lie algebras $\mathcal{G}$ (ADE algebras), we have $h_{\mathcal{G}} = h_{\mathcal{G}}^{\vee}$, and the preceding scaling limits apply. However, for non simply-laced algebras (B_n, C_n, G_2, F_4), we have $h_{\mathcal{G}} > h_{\mathcal{G}}^{\vee}$, and our earlier considerations show that the untwisted elliptic Calogero-Moser Hamiltonians do not tend to a finite limit under (4.6), $q \to 0$, M is kept fixed. This is why the twisted Hamiltonian systems (3.6) have to be introduced. The twisting produces precisely an improvement in the asymptotic behavior of the potential which allows a finite, non-trivial limit. More precisely, we can write

$$m^2 \wp_{\nu}(x) = \frac{\nu^2}{2} \sum_{n=-\infty}^{\infty} \frac{m^2}{\mathrm{ch}\,\nu(x - 2n\omega_2) - 1}. \tag{4.8}$$

Setting $x = X - 2\omega_2 \delta^{\vee}\rho$, we obtain the following asymptotics

$$m^2 \wp_{\nu}(x) = \nu^2 M^2 \begin{cases} e^{-2\omega_2(\delta^{\vee}\alpha^{\vee}\cdot\rho - \delta^{\vee}) - \alpha^{\vee}\cdot X} + e^{-2\omega_2(1-\delta^{\vee}\alpha^{\vee}\cdot\rho - \delta^{\vee}) + \alpha^{\vee}\cdot X}, & \text{if } \alpha \text{ is long;} \\ e^{-2\omega_2(\delta^{\vee}\alpha^{\vee}\cdot\rho - \delta^{\vee}) - \alpha^{\vee}\cdot X}, & \text{if } \alpha \text{ is short.} \end{cases}$$

This leads to the following theorem [8]

Theorem 2. *Under the limit* $x = X + 2\omega_2 \frac{1}{h_{\mathcal{G}}^{\vee}}\rho$, $m = M q^{-\frac{1}{2h_{\mathcal{G}}^{\vee}}}$, *with* ρ *the Weyl vector and* $q \to 0$, *the Hamiltonian of the twisted elliptic Calogero-Moser system for the simple Lie algebra* $\mathcal{G}$ *tends to the Hamiltonian of the Toda system for the affine Lie algebra* $(\mathcal{G}^{(1)})^{\vee}$.

Similar arguments show that the Lax pairs constructed below also have finite, non-trivial scaling limits whenever this is the case for the Hamiltonians.

V. LAX PAIRS FOR CALOGERO-MOSER SYSTEMS

a) The General Ansatz

Let the rank of $\mathcal{G}$ be n, and d be its dimension. Let Λ be a representation of $\mathcal{G}$ of dimension N, of weights λ_I, $1 \leq I \leq N$. Let $u_I \in \mathbf{C}^N$ be the weights of the fundamental representation of $GL(N,\mathbf{C})$. Project orthogonally the u_I's onto the λ_I's as

$$su_I = \lambda_I + u_I, \qquad \lambda_I \perp v_J. \tag{5.1}$$

It is easily verified that s^2 is the second Dynkin index. Then $\alpha_{IJ} = \lambda_I - \lambda_J$ is a weight of $\Lambda \otimes \Lambda^*$ associated to the root $u_I - u_J$ of $GL(N,\mathbf{C})$. The Lax pairs for both untwisted and twisted Calogero-Moser systems will be of the form

$$L = P + X, \quad M = D + X, \tag{5.2}$$

where the matrices P, X, D, and Y are given by

$$X = \sum_{I \neq J} C_{IJ} \Phi_{IJ}(\alpha_{IJ}, z) E_{IJ}, \quad Y = \sum_{I \neq j} C_{IJ} \Phi'_{IJ}(\alpha_{IJ}, z) E_{IJ} \tag{5.3a}$$

and by

$$P = p \cdot h, \qquad D = d \cdot (h \oplus \tilde{h}) + \Delta. \tag{5.3b}$$

Here h is in a Cartan subalgebra $\mathcal{H}_\mathcal{G}$ for $\mathcal{G}$, $\tilde{h}$ is in the Cartan-Killing orthogonal complement of $\mathcal{H}_\mathcal{G}$ inside a Cartan subalgebra $\mathcal{H}$ for $GL(N,\mathbf{C})$, and Δ is in the centralizer of $\mathcal{H}_\mathcal{G}$ in $GL(N,\mathbf{C})$. The functions $\Phi_{IJ}(x,z)$ and the coefficients C_{IJ} are yet to be determined. We begin by stating the necessary and sufficient conditions for the pair $L(z)$, $M(z)$ of (5.2) to be a Lax pair for the (twisted or untwisted) Calogero-Moser systems. For this, it is convenient to introduce the following notation

$$\begin{aligned}
\Phi_{IJ} &= \Phi_{IJ}(\alpha_{IJ} \cdot x) \\
\wp'_{IJ} &= \Phi_{IJ}(\alpha_{IJ} \cdot x, z) \Phi'_{JI}(-\alpha_{IJ} \cdot x, z) - \Phi_{IJ}(-\alpha_{IJ} \cdot x, z) \Phi'_{JI}(\alpha_{IJ} \cdot x, z).
\end{aligned} \tag{5.4}$$

Then the Lax equation $\dot{L}(z) = [L(z), M(z)]$ implies the Calogero-Moser system if and only if the following three identities are satisfied

$$\sum_{I \neq J} C_{IJ} C_{JI} \wp'_{IJ} \alpha_{IJ} = s^2 \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m^2_{|\alpha|} \wp_{\nu(\alpha)}(\alpha \cdot x) \tag{5.5a}$$

$$\sum_{I \neq J} C_{IJ} C_{JI} \wp'_{IJ} (v_I - v_J) = 0 \tag{5.5b}$$

$$\sum_{K\neq I,J} C_{IK}C_{KJ}(\Phi_{IK}\Phi'_{KJ} - \Phi'_{IK}\Phi_{KJ}) = sC_{IJ}\Phi_{IJ}d\cdot(v_I - v_J) + \sum_{K\neq I,J}\Delta_{IJ}C_{KJ}\Phi_{KJ} - \sum_{K\neq I,J} C_{IK}\Phi_{IK}\Delta_{KJ} \tag{5.5c}$$

The following theorem was established in [7]:

Theorem 3. *A representation Λ, functions Φ_{IJ}, and coefficients C_{IJ} with a spectral parameter z satisfying (5.5a-c) can be found for all twisted and untwisted elliptic Calogero-Moser systems associated with a simple Lie algebra $\mathcal{G}$, except possibly in the case of twisted G_2. In the case of E_8, we have to assume the existence of a ± 1 cocycle.*

b) Lax Pairs for Untwisted Calogero-Moser Systems

We now describe some important features of the Lax pairs we obtain in this manner. All have an independent spectral parameter.

- In the case of the *untwisted* Calogero-Moser systems, we can choose $\Phi_{IJ}(x,z) = \Phi(x,z)$, $\wp_{IJ}(x) = \wp(x)$ for all $\mathcal{G}$. One also has $\Delta = 0$ for all $\mathcal{G}$, except for E_8.

- For A_n, the Lax pair (5.2-5.3) may correspond to any of the totally antisymmetric representations, including the fundamental one.

- For the BC_n system, the Lax pair is obtained by imbedding B_n in $GL(N, \mathbf{C})$ with $N = 2n+1$. When $z = \omega_a$ (half-period), the Lax pair obtained this way reduces to that of Olshanetsky and Perelomov [6].

- For the B_n and D_n systems, additional Lax pairs are found by taking Λ to be the spinor representation.

- For G_2, a Lax pair is obtained in the representation $\mathbf{7}$ while another one is gotten by restricting the $\mathbf{8}$ of B_3 to the $\mathbf{7}\oplus\mathbf{1}$ of G_2.

- For F_4, a Lax pair can be obtained by taking Λ to be the $\mathbf{26}\oplus\mathbf{1}$ of F_4, viewed as the restriction of the $\mathbf{27}$ of E_6 to its F_4 subalgebra.

- For E_6, Λ is the $\mathbf{27}$ representation.

- For E_7, Λ is the $\mathbf{56}$ representation.

- For E_8, a Lax pair with spectral parameter can be constructed with Λ given by the $\mathbf{248}$ representation, if coefficients $c_{IJ} = \pm 1$ exist with the following cocycle conditions

$$c(\lambda, \lambda-\delta)c(\lambda-\delta, \mu) = c(\lambda, \mu+\delta)c(\mu+\delta, \mu) \qquad \text{when } \delta\cdot\lambda = -\delta\cdot\mu = 1,\ \lambda\cdot\mu = 0$$

$$
\begin{aligned}
c(\lambda,\mu)c(\lambda-\delta,\mu) =& c(\lambda,\lambda-\delta) \\
& \text{when } \delta\cdot\lambda=\lambda\cdot\mu=1,\ \delta\cdot\mu=0 \\
c(\lambda,\mu)c(\lambda,\lambda-\mu) =& -c(\lambda-\mu,-\mu) \\
& \text{when } \lambda\cdot\mu=1.
\end{aligned}
\tag{5.6a}
$$

The matrix Δ in the Lax pair is then the 8×8 matrix given by

$$
\begin{aligned}
\Delta_{ab} =& \sum_{\substack{\delta\cdot\beta_a=1\\ \delta\cdot\beta_b=1}} \frac{m_2}{2}\big(c(\beta_a,\delta)c(\delta,\beta_b)+c(\beta_a,\beta_a-\delta)c(\beta_a-\delta,\beta_b)\big)\wp(\delta\cdot x) \\
& - \sum_{\substack{\delta\cdot\beta_a=1\\ \delta\cdot\beta_b=-1}} \frac{m_2}{2}\big(c(\beta_a,\delta)c(\delta,\beta_b)+c(\beta_a,\beta_a-\delta)c(\beta_a-\delta,\beta_b)\big)\wp(\delta\cdot x) \\
\Delta_{aa} =& \sum_{\beta_a\cdot\delta=1} m_2\wp(\delta\cdot x)+2m_2\wp(\beta_a\cdot x),
\end{aligned}
\tag{5.6b}
$$

where β_a, $1\le a\le 8$, is a maximal set of 8 mutually orthogonal roots.

• Explicit expressions for the constants C_{IJ} and the functions $d(x)$, and thus for the Lax pair are particularly simple when the representation Λ consists of only a single Weyl orbit of weights. This is the case when Λ is either

(1) the defining representation of A_n, C_n or D_n;
(2) any rank p totally anti-symmetric representation of A_n;
(3) an irreducible fundamental spinor representation of B_n or D_n;
(4) the **27** of E_6;
(5) the **56** of E_7.

Then the weights λ and μ of Λ provide unique labels instead of I and J, and the values of $C_{IJ}=C_{\lambda\mu}$ are given by a simple formula

$$
C_{\lambda\mu} = \begin{cases} \sqrt{\frac{\alpha^2}{2}}m_{|\alpha|} & \text{when } \alpha=\lambda-\mu \text{ is a root} \\ 0 & \text{otherwise} \end{cases}
\tag{5.7}
$$

The expression for the vector d may be summarized by

$$
sd\cdot u_\lambda = \sum_{\lambda\cdot\delta=1;\ \delta^2=2} m_{|\delta|}\wp(\delta\cdot x)
\tag{5.8}
$$

(For C_n, the last equation has an additional term, as given in [7].) In each case, the number of independent couplings $m_{|\alpha|}$ equals the number of different root lengths.

c) Lax Pairs for Twisted Calogero-Moser Systems

Recall that the twisted and untwisted Calogero-Moser systems differ only for non-simply laced Lie algebras, namely B_n, C_n, G_2 and F_4. These are the only algebras we discuss in this paragraph. The construction (5.2-5.5) gives then Lax pairs for all of them, with the possible exception of twisted G_2. Unlike the case of untwisted Lie algebras however, the functions Φ_{IJ} have to be chosen with care, and differ for each algebra. More specifically,

• For B_n, the Lax pair is of dimension $N = 2n$, admits two independent couplings m_1 and m_2, and

$$\Phi_{IJ}(x,z) = \begin{cases} \Phi(x,z), & \text{if } I - J \neq 0, \pm n \\ \Phi_2(\frac{1}{2}x, z), & \text{if } I - J = \pm n \end{cases} . \tag{5.9}$$

Here a new function $\Phi_2(x,z)$ is defined by

$$\Phi_2(\frac{1}{2}x, z) = \frac{\Phi(\frac{1}{2}x, z)\Phi(\frac{1}{2}x + \omega_1, z)}{\Phi(\omega_1, z)} \tag{5.10}$$

• For C_n, the Lax pair is of dimension $N = 2n+2$, admits one independent coupling m_2, and

$$\Phi_{IJ}(x,z) = \Phi_2(x + \omega_{IJ}, z),$$

where ω_{IJ} are given by

$$\omega_{IJ} = \begin{cases} 0, & \text{if } I \neq J = 1, 2, \cdots, 2n+1; \\ \omega_2, & \text{if } 1 \leq I \leq 2n, \ J = 2n+2; \\ -\omega_2, & \text{if } 1 \leq J \leq 2n, \ I = 2n+2. \end{cases} \tag{5.11}$$

• For F_4, the Lax pair is of dimension $N = 24$, admits two independent couplings m_1 and m_2,

$$\Phi_{\lambda\mu}(x,z) = \begin{cases} \Phi(x,z), & \text{if } \lambda \cdot \mu = 0; \\ \Phi_1(x,z), & \text{if } \lambda \cdot \mu = \frac{1}{2}; \\ \Phi_2(\frac{1}{2}x, z), & \text{if } \lambda \cdot \mu = -1. \end{cases} \tag{5.12}$$

where the function $\Phi_1(x,z)$ is defined by

$$\Phi_1(x,z) = \Phi(x,z) - e^{\pi i \zeta(z) + \eta_1 z}\Phi(x + \omega_1, z) \tag{5.13}$$

Here it is more convenient to label the entries of the Lax pair directly by the weights $\lambda = \lambda_I$ and $\mu = \lambda_J$ instead of I and J.

• For G_2, there are natural candidates for Lax pairs in the **6** and **8** representations of G_2, but it is still unknown whether elliptic functions $\Phi_{IJ}(x,z)$ exist which satisfy the required identities.

We note that recently Lax pairs of root type have been considered [21] which correspond, in the above Ansatz (5.3-5), to Λ equal to the adjoint representation of $\mathcal{G}$ and the coefficients C_{IJ} vanishing for I or J associated with zero weights. This choice yields another Lax pair for the case of E_8.

VI. CALOGERO-MOSER AND SEIBERG-WITTEN THEORY

The correspondence between Seiberg-Witten theory for $\mathcal{N}=2$ super-Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge algebra, and the elliptic Calogero-Moser systems was first established in [5], for the gauge algebra $\mathcal{G}=SU(N)$. We describe it here in some detail.

a) The Case of $\mathcal{G}=SU(N)$

All that we shall need here of the elliptic Calogero-Moser system is its Lax operator $L(z)$, whose $N\times N$ matrix elements are given by

$$L_{ij}(z)=p_i\delta_{ij}-m(1-\delta_{ij})\Phi(x_i-x_j,z) \tag{6.1}$$

Notice that the Hamiltonian is simply given in terms of L by $H(x,p)=\frac{1}{2}\mathrm{tr}L(z)^2+C\wp(z)$ with $C=-\frac{1}{2}m^2N(N-1)$. The correspondence between the data of the elliptic Calogero-Moser system and those of the Seiberg-Witten theory is as follows.

(1) The parameter m in (6.1) is the hypermultiplet mass;

(2) The gauge coupling g and the θ-angle are related to the modulus of the torus $\Sigma=\mathbf{C}/(2\omega_1\mathbf{Z}+2\omega_2\mathbf{Z})$ by

$$\tau=\frac{\omega_2}{\omega_1}=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}\,; \tag{6.2}$$

(3) The Seiberg-Witten curve Γ is the spectral curve of the elliptic Calogero-Moser model,

$$\Gamma=\{(k,z)\in\mathbf{C}\times\Sigma,\ \det\big(kI-L(z)\big)=0\} \tag{6.3}$$

The Seiberg-Witten form is $d\lambda=k\,dz$. Γ is invariant under the Weyl group of $SU(N)$.

(4) Using the Lax equation $\dot{L}=[L,M]$, it is clear that the spectral curve is independent of time, and can be dependent only upon the constants of motion of the Calogero-Moser system, of which there are only N. These integrals of motion may be viewed as parametrized by the quantum moduli of the Seiberg-Witten system.

(5) Finally, $d\lambda=kdz$ is meromorphic, with a simple pole on each of the N sheets above the point $z=0$ on the base torus. The residue at each of these poles is proportional to m, as required by the general set-up of Seiberg-Witten theory, explained in §II.

b) Four Fundamental Results for the Case of $\mathcal{G}=SU(N)$

While the above mappings of the Seiberg-Witten data onto the Calogero-Moser data is certainly natural, there is no direct proof of it, and it is important to check that the results inferred from it agree with known facts from quantum field theory. To establish this, as well as a series of further predictions from the correspondence, we give four theorems (the proofs may be found in [5] for the first three theorems, and in [29] for the last one).

Theorem 4. *The spectral curve equation* $\det(kI - L(z)) = 0$ *is equivalent to*

$$\vartheta_1\left(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau\right)H(k) = 0 \tag{6.4}$$

where $H(k)$ *is a monic polynomial in* k *of degree* N*, whose zeros (or equivalently whose coefficients) correspond to the moduli of the gauge theory. If* $H(k) = \prod_{i=1}^{N}(k - k_i)$*, then*

$$\lim_{q\to 0}\frac{1}{2\pi i}\oint_{A_i} kdz = k_i - \frac{1}{2}m.$$

Here, ϑ_1 is the Jacobi ϑ-function, which admits a simple series expansion in powers of the instanton factor $q = e^{2\pi i\tau}$, so that the curve equation may also be rewritten as a series expansion

$$\sum_{n\in\mathbf{Z}}(-)^n q^{\frac{1}{2}n(n-1)}e^{nz}H(k - n\cdot m) = 0 \tag{6.5}$$

where we have set $\omega_1 = -i\pi$ without loss of generality.

Theorem 5. *The prepotential of the Seiberg-Witten theory obeys a renormalization group-type equation that simply relates* $\mathcal{F}$ *to the Calogero-Moser Hamiltonian, expressed in terms of the quantum order parameters* a_j

$$a_j = \frac{1}{2\pi i}\oint_{A_j} d\lambda \qquad\qquad \left.\frac{\partial\mathcal{F}}{\partial\tau}\right|_{a_j} = H(x,p) = \frac{1}{2}\mathrm{tr}L(z)^2 + C\wp(z) \tag{6.6}$$

Furthermore, in an expansion in powers of the instanton factor $q = e^{2\pi i\tau}$*, the quantum order parameters* a_j *may be computed by residue methods in terms of the zeros of* $H(k)$*.*

The proof of (6.6) requires Riemann surface deformation theory [5]. The fact that the quantum order parameters may be evaluated by residue methods arises from the fact that A_j-cycles may be chosen on the spectral curve Γ in such a way that they will shrink to zero as $q \to 0$. As a result, contour integrals around full-fledged branch cuts A_j reduce to contour integrals around poles at single points, which may be calculated by residue methods only. These methods were originally developed in [30,31]. Knowing the quantum order parameters in terms of the zeros k_j of $H(k) = 0$ is a relation which may be inverted and used in (6.6) to obtain a differential relation for all order instanton corrections. It is now only necessary to evaluate explicitly the τ-independent contribution to $\mathcal{F}$, which in field theory arises from perturbation theory. This may be done easily by retaining only the $n=0$ and $n=1$ terms in the expansion of the curve (6.5), so that $z=\ln H(k)-\ln H(k-m)$. The results of the calculations to two instanton order may be summarized in the following theorem [5].

Theorem 6. *The prepotential, to 2 instanton order is given by* $\mathcal{F}=\mathcal{F}^{(\mathrm{pert})}+\mathcal{F}^{(1)}+\mathcal{F}^{(2)}$. *The perturbative contribution is given by*

$$\mathcal{F}^{(\mathrm{pert})}=\frac{\tau}{2}\sum_i a_i^2-\frac{1}{8\pi i}\sum_{i,j}\Big[(a_i-a_j)^2\ln(a_i-a_j)^2-(a_i-a_j-m)^2\ln(a_i-a_j-m)^2\Big] \tag{6.7a}$$

while all instanton corrections are expressed in terms of a single function

$$S_i(a)=\frac{\prod_{j=1}^N\left[(a_i-a_j)^2-m^2\right]}{\prod_{j\neq i}(a-a_j)^2} \tag{6.7b}$$

as follows

$$\begin{aligned}\mathcal{F}^{(1)}&=\frac{q}{2\pi i}\sum_i S_i(a_i)\\ \mathcal{F}^{(2)}&=\frac{q^2}{8\pi i}\Big[\sum_i S_i(a_i)\partial_i^2S_i(a_i)+4\sum_{i\neq j}\frac{S_i(a_i)S_j(a_j)}{(a_i-a_j)^2}-\frac{S_i(a_i)S_j(a_j)}{(a_i-a_j-m)^2}\Big]\end{aligned} \tag{6.7c}$$

The perturbative corrections to the prepotential of (6.7a) indeed precisely agree with the predictions of asymptotic freedom. The formulas (6.7c) for the instanton corrections $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are new, as they have not yet been computed by direct field theory methods. Perturbative expansions of the prepotential in powers of m have also been obtained in [32]. Within the context of topological gauge theory, these results on the prepotential have been used in [33].

The moduli k_i, $1\leq i\leq N$, of the gauge theory are evidently integrals of motion of the system. To identify these integrals of motion, denote by S any subset of $\{1,\cdots,N\}$, and let $S^*=\{1,\cdots,N\}\setminus S$, $\wp(S)=\wp(x_i-x_j)$ when $S=\{i,j\}$. Let also p_S denote the subset of momenta p_i with $i\in S$, and $\sigma_k(p_S)$ the k-th symmetric polynomial in p_i, $i\in S$. We have [29]

Theorem 7. *For any* K, $0\leq K\leq N$, *let* $\sigma_K(k_1,\cdots,k_N)=\sigma_K(k)$ *be the* K*-th symmetric polynomial of* $(k_1,\cdots,k_N)$, *defined by* $H(u)=\sum_{K=0}^N(-)^K\sigma_K(k)u^{N-K}$. *Then*

$$\sigma_K(k)=\sigma_K(p)+\sum_{l=1}^{[K/2]}m^{2l}\sum_{\substack{|S_i\cap S_j|=2\delta_{ij}\\1\leq i,j\leq l}}\sigma_{K-2l}(p_{(\cup_{i=1}^lS_i)^*})\prod_{i=1}^l[\wp(S_i)+\frac{\eta_1}{\omega_1}] \tag{6.8}$$

We note that an alternative derivation of (6.4) was recently presented in [34]. The parametrization (6.4) has also been extended to the spectral curves of spin Calogero-Moser systems in [10]. For spin Calogero-Moser systems with l internal degrees of freedom, we can write

$$det(\lambda I - L(z)) = \sum_{p=1}^{l} \partial_z^{p-1} \Big(\frac{\theta_1(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau)}{\theta_1(\frac{z}{2\omega_1}|\tau)} H_p(k) \Big)_{|k=\lambda + m\partial_z \log\theta_1(\frac{z}{2\omega_1}|\tau)}, \tag{6.9}$$

where $H_p(k)$ is a polynomial of degree $N - p + 1$, $1 \leq p \leq l$, $H_1(k)$ is monic, and $H_p(k)$ has no term of order k^0.

c) The Case of General Gauge Algebra $\mathcal{G}$

We consider now the $\mathcal{N} = 2$ supersymmetric gauge theory for a general simple gauge algebra $\mathcal{G}$ and a hypermultiplet of mass m in the adjoint representation. Then [9]

• the Seiberg-Witten curve of the theory is given by the spectral curve $\Gamma = \{(k, z) \in \mathbf{C} \times \Sigma; \det(kI - L(z)) = 0\}$ of the *twisted* elliptic Calogero-Moser system associated to the Lie algebra $\mathcal{G}$. The Seiberg-Witten differential $d\lambda$ is given by $d\lambda = kdz$.

• The function $R(k, z) = \det(kI - L(z))$ is polynomial in k and meromorphic in z. The spectral curve Γ is invariant under the Weyl group of $\mathcal{G}$. It depends on n complex moduli, which can be thought of as independent integrals of motion of the Calogero-Moser system.

• The differential $d\lambda = kdz$ is meromorphic on Γ, with simple poles. The position and residues of the poles are independent of the moduli. The residues are linear in the hypermultiplet mass m. (Unlike the case of $SU(N)$, their exact values are difficult to determine for general $\mathcal{G}$).

• In the $m \to 0$ limit, the Calogero-Moser system reduces to a free system, the spectral curve Γ is just the product of several unglued copies of the base torus Σ, indexed by the constant eigenvalues of $L(z) = p \cdot h$. Let k_i, $1 \leq i \leq n$, be n independent eigenvalues, and A_i, B_i be the A and B cycles lifted to the corresponding sheets. For each i, we readily obtain

$$\begin{aligned} a_i &= \frac{1}{2\pi i} \oint_{A_i} d\lambda = \frac{k_i}{2\pi i} \oint_A dz = \frac{2\omega_1}{2\pi i} k_i \\ a_{Di} &= \frac{1}{2\pi i} \oint_{B_i} d\lambda = \frac{k_i}{2\pi i} \oint_B dz = \frac{2\omega_1}{2\pi i} \tau k_i \end{aligned} \tag{6.9}$$

Thus the prepotential $\mathcal{F}$ is given by $\mathcal{F} = \frac{\tau}{2} \sum_{i=1}^n a_i^2$. This is the classical prepotential and hence the correct answer, since in the $m \to 0$ limit, the theory acquires an $\mathcal{N} = 4$ supersymmetry, and receives no quantum corrections.

• The $m \to \infty$ limit is the crucial consistency check, which motivated the introduction of the *twisted* Calogero-Moser systems in the first place [7,8]. In view of Theorem 2 and subsequent comments, in the limit $m \to \infty$, $q \to 0$, with $x = X + 2\omega_2 \frac{1}{h_{\mathcal{G}}^\vee}\rho$, $m = Mq^{-\frac{1}{2h_{\mathcal{G}}^\vee}}$ with X and M kept fixed, the Hamiltonian and spectral curve for the twisted elliptic Calogero-Moser system with Lie algebra $\mathcal{G}$ reduce to the Hamiltonian and spectral curve for the Toda system for the affine Lie algebra $(\mathcal{G}^{(1)})^\vee$. This is the correct answer. Indeed, in this limit, the gauge theory with adjoint hypermultiplet reduces to the pure Yang-Mills theory, and the Seiberg-Witten spectral curves for pure Yang-Mills with gauge algebra $\mathcal{G}$ have been shown by Martinec and Warner [35] to be the spectral curves of the Toda system for $(\mathcal{G}^{(1)})^\vee$.

• The equations $R(k,z) = det(kI - L(z))$ for the spectral curves of $\mathcal{G} = D_n$ can be written explicitly in the trigonometric limit $\tau \to i\infty$, in terms of a polynomial $H(A) \equiv \prod_{j=1}^n (A^2 - k_j^2)$ similar to (6.4)

$$R(k,z) = \frac{m^2 + mA - 2k\frac{m}{Z}}{m^2 + 2mA} H(A) + \frac{mA + 2k\frac{m}{Z}}{m^2 + 2mA} H(A+m), \tag{6.10}$$

where we have introduced the more convenient spectral parameter Z by $\frac{1}{Z} = \frac{1}{2}\coth\frac{z}{2}$, and the variable A is defined by the quadratic relation

$$A^2 + mA + 2k\frac{m}{Z} - k^2 = 0.$$

The effective prepotential can be evaluated explicitly in the case of $\mathcal{G} = D_n$ for $n \leq 5$. Its logarithmic singularity does reproduce the logarithmic singularities expected from field theory considerations.

• As in the known correspondences between Seiberg-Witten theory and integrable models [5,30], we expect the following equation to hold

$$\frac{\partial \mathcal{F}}{\partial \tau} = H_{\mathcal{G}}^{twisted}(x,p), \tag{6.11}$$

to hold. Note that the left hand side can be interpreted in the gauge theory as a renormalization group equation.

• For simply-laced $\mathcal{G}$, the curves $R(k,z) = 0$ are modular invariant. Physically, the gauge theories for these Lie algebras are self-dual. For non simply-laced $\mathcal{G}$, the modular group is broken to the congruence subgroup $\Gamma_0(2)$ for $\mathcal{G} = B_n, C_n$, F_4, and to $\Gamma_0(3)$ for G_2. The Hamiltonians of the twisted Calogero-Moser systems for non-simply laced $\mathcal{G}$ are also transformed under Landen transformations into the Hamiltonians of the twisted Calogero-Moser system for the dual algebra $\mathcal{G}^\vee$. It would be interesting to determine whether such transformations exist for the spectral curves or the corresponding gauge theories themselves.

Spectral curves for certain gauge theories with classical gauge algebras and matter in the adjoint representation have also been proposed in [15] (see in particular the papers by Witten, Uranga, and Yokono), based on branes in string theory and M-theory. Connections between branes configurations associated with $\mathcal{N}=2$ gauge theories and integrable systems have been put forward in [36].

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REFERENCES

[1] N. Seiberg and E. Witten, "Electro-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory", Nucl. Phys. **B 426** (1994) 19-53, hep-th/9407087;
N. Seiberg and E. witten, "Monopoles, duality, and chiral symmetry breaking in $\mathcal{N}=2$ supersymmetric QCD", Nucl. Phys. **B 431** (1994) 494, hep-th/9410167.

[2] A. Gorskii, I.M. Krichever, A. Marshakov, A. Mironov and A. Morozov, "Integrability and Seiberg-Witten exact solution", Phys. Lett. **B355** (1995) 466, hep-th/9505035.

[3] R. Donagi and E. Witten, "Supersymmetric Yang-Mills and integrable systems", Nucl. Phys. **B 460** (1996) 288-334, hep-th/9510101.

[4] A. Gorsky and N. Nekrasov, "Elliptic Calogero-Moser System from two-dimensional current algebra", hep-th/9401021;
N. Nekrasov, "Holomorphic bundles and many-body systems", Comm. Math. Phys. **180** (1996) 587;
E. Martinec, "Integrable structures in supersymmetric gauge and string theory", hep-th/9510204.

[5] E. D'Hoker and D.H. Phong, "Calogero-Moser systems in $SU(N)$ Seiberg-Witten theory", Nucl. Phys. **B 513** (1998) 405-444, hep-th/9709053.

[6] M.A. Olshanetsky and A.M. Perelomov, "Completely integrable Hamiltonian systems connected with semisimple Lie algebras", Inventiones Math. **37** (1976) 93-108;
M.A. Olshanetsky and A.M. Perelomov, "Classical integrable finite-dimensional systems related to Lie algebras", Phys. Rep. **71 C** (1981) 313-400.

[7] E. D'Hoker and D.H. Phong, "Calogero-Moser Lax pairs with spectral parameter for general Lie algebras", Nucl. Phys. **B 530** (1998) 537-610, hep-th/9804124.

[8] E. D'Hoker and D.H. Phong, "Calogero-Moser and Toda systems for twisted and untwisted affine Lie algebras", Nucl. Phys. **B 530** (1998) 611-640, hep-th/9804125.

[9] E. D'Hoker and D.H. Phong, "Spectral curves for super Yang-Mills with adjoint hypermultiplet for general Lie algebras", Nucl. Phys. **B 534** (1998) 697-719, hep-th/9804126.

[10] E. D'Hoker and D.H. Phong, "Lax Pairs and Spectral Curves for Calogero-Moser and Spin Calogero-Moser Systems", hep-th/9903002, to appear in *Regular and Chaotic Dynamics*;
E. D'Hoker and D.H. Phong, "Seiberg-Witten Theory and Integrable Systems", hep-th/9903068, to appear in *Proceedings of the Edinburgh Conference on Seiberg-Witten and Whitham theories*, ed. by H. Braden and I.M. Krichever.

[11] W. Lerche, "Introduction to Seiberg-Witten theory and its stringy origins", Proceedings of the *Spring School and Workshop on String Theory*, ICTP, Trieste (1996), hep-th/9611190, Nucl. Phys. Proc. Suppl. **B 55** (1997) 83;
Y. Ohta, "Non-perturbative solutions in N=2 supersymmetric Yang-Mills theories: progress and perspective", hep-th/9903182.

[12] I.M. Krichever and D.H. Phong, "Symplectic forms in the theory of solitons", hep-th/9708170, to appear in Surveys in Differential Geometry, Vol. III
I.M. Krichever and D.H. Phong, "On the integrable geometry of $N = 2$ supersymmetric gauge theories and soliton equations", J. Differential Geometry **45** (1997) 349-389.

[13] R. Donagi, "Seiberg-Witten integrable models", alg-geom/9705010;
R. Carroll, "Prepotentials and Riemann surfaces", hep-th/9802130;
R. Carroll, "Remarks on Whitham and RG", hep-th/9712110,
D. Freed, "Special Kähler manifolds", hep-th/9712042

[14] A. Marshakov, "On integrable systems and supersymmetric gauge theories", Theor. Math. Phys. **112** (1997) 791-826, hep-th/9702083. A. Marshakov, "Seiberg-Witten curves and integrable systems", hep-th/9903252;
A. Mironov, "WDVV equations and Seiberg-Witten theories", hep-th/9903088.

[15] E. Witten, "Solutions of four-dimensional field theories via M Theory", Nucl. Phys. **B 500** (1997) 3, hep-th/9703166;
A. Brandhuber, J. Sonnenschein, S. Theisen, and S. Yankielowicz, "M-Theory and Seiberg-Witten curves: orthogonal and symplectic groups", Nucl. Phys. **B 504** (1997) 175, hep-th/9705232;
A. Giveon and D. Kutasov, "Brane dynamics and gauge theory", hep-th/9802067 K. Landsteiner, E. Lopez, and D. Lowe, "New curves from branes", Nucl. Phys. **B 516** (1998) 273, hep-th/9708118;

A.M. Uranga, "Towards mass deformed $\mathcal{N} = 4$ $SO(N)$ and $Sp(K)$ gauge theories from brane configurations", Nucl. Phys. **B 526** (1998) 241-277, hep-th/9803054;
T. Yokono, "Orientifold four plane in brane configurations and $\mathcal{N} = 4$ $USp(2N)$ and $SO(2N)$ theory", Nucl. Phys. **B 532** (1998) 210-226, hep-th/9803123;
K. Landsteiner, E. Lopez, and D. Lowe, "Supersymmetric gauge theories from branes and orientifold six-planes", hep-th/9805158.

[16] S. Kachru and C. Vafa, "Exact results for N=2 compactifications of heterotic strings", Nucl. Phys. **B 450** (1995) 69, hep-th/9505105;
M. Bershadsky, K. Intriligator, S. Kachru, D. Morrison, V. Sadov, and C. Vafa, "Geometric singularities and enhanced gauge symmetries", Nucl. Phys. **B 481** (1996) 215, hep-th/9605200;
S. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories", Nucl. Phys. **B 497** (1997) 173, hep-th/9609239;
S. Katz, P. Mayr, and C. Vafa, "Mirror symmetry and exact solutions of 4D $\mathcal{N} = 2$ gauge theories", Adv. Theor. Math. Phys. **1** (1998) 53, hep-th/9706110

[17] I.P. Ennes, S.G. Naculich, H. Rhedin, and H. Schnitzer, "One-instanton predictions of a Seiberg-Witten curve from M-Theory: the symmetric representation", Nucl. Phys. **B 533** (1998) 275-302, hep-th/9804105;
I.P. Ennes, S.G. Naculich, H. Rhedin, and H. Schnitzer, "One-instanton predictions of a Seiberg-Witten curve from M-Theory: the anti-symmetric representation", Int. J. Mod. Phys. **A 14** (1999) 301-321; hep-th/9804151;
I.P. Ennes, S.G. Naculich, H. Rhedin, and H. Schnitzer, "One-instanton predictions for non-hyperelliptic curves derived from M-Theory", Nucl.Phys. **B536** (1998) 245-257, hep-th/9806144;
I.P. Ennes, S.G. Naculich, H. Rhedin, and H. Schnitzer, "One-instanton predictions of Seiberg-Witten curves for product groups", hep-th/9901124;
I.P. Ennes, S.G. Naculich, H. Rhedin, and H. Schnitzer, "Two antisymmetric hypermultiplets in N=2 SU(N) gauge theory: Seiberg-Witten curve and M Theory interpretation", hep-th/9904078.

[18] I.M. Krichever, "The τ-function of the universal Whitham hierarchy, matrix models, and topological field theories", Comm. Pure Appl. Math. **47** (1994) 437-475;
T. Eguchi and S.K. Yang, "Prepotentials of N=2 supersymmetric gauge theories and soliton equations", hep-th/9510183;
T. Nakatsu and K. Takasaki, "Whitham-Toda hierarchy and N=2 supersymmetric Yang-Mills theory", Mod. Phys. Lett. **A 11** (1996) 157-168, hep-th/9509162;
K. Takasaki, "Whitham deformations and tau functions in N=2 supersymmetric gauge theories", hep-th/9905224;
J. Edelstein, M. Gomez-Reina, and J. Mas, "Instanton corrections in N=2 supersym-

metric theories with classical gauge groups and fundamental matter hypermultiplets", hep-th/9904087.

[19] G. Bonelli and M. Matone, "Nonperturbative relations in N=2 SUSY Yang-Mills and WDVV equations", Phys. Rev. Lett. **77** (1996) 4712, hep-th/9606090;
A. Marshakov, A. Mironov, and A. Morozov, "WDVV-like equations in N=2 SUST Yang-Mills theory", Phys. Lett. **B 389** (1996) 43, hep-th/9607109;
J.M. Isidro, "On the WDVV equation and M-Theory", Nucl. Phys. **B 539** (1999) 379-402.

[20] F. Calogero, "Exactly solvable one-dimensional many-body problems", Lett. Nuovo Cim. **13** (1975) 411-416.

[21] J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations", Advances Math. **16** (1975) 197.

[22] H. Braden, "A conjectured R-matrix", J. Phys. A **31** (1998) 1733-1741.

[23] A.P. Polychronakos, "Generalized Calogero-Sutherland systems from many matrix models", Nucl. Phys. **B 546** (1999) 495-502;
A.P. Polychronakos, "Generalized Statistics in one dimension", hep-th/9902157.

[24] I.M. Krichever, "Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles", Funct. Anal. Appl. **14** (1980) 282-290.

[25] H.W. Braden and V.M. Buchstaber, "The general analytic solution of a functional equation of addition type", Siam J. Math. anal. **28** (1997) 903-923.

[26] A. Bordner, E. Corrigan, and R. Sasaki, "Calogero-Moser systems: a new formulation", hep-th/9805106;
A. Bordner, R. Sasaki, and K. Takasaki, "Calogero-Moser systems II: symmetries and foldings", hep-th/9809068;
A. Bordner and R. Sasaki, "Calogero-Moser systems III: Elliptic potentials and twisting", hep-th/9812232;
A. Bordner, E. Corrigan, and R. Sasaki, "Generalized Calogero-Moser models and universal Lax pair operators", hep-th/9905011.

[27] I. Inozemtsev, "Lax representation with spectral parameter on a torus for integrable particle systems", Lett. Math. Phys. **17** (1989) 11-17;
I. Inozemtsev, "The finite Toda lattices", Comm. Math. Phys. **121** (1989) 628-638.

[28] P. Goddard and D. Olive, "Kac-Moody and Virasoro algebras in relation to quantum physics", International J. Mod. Phys. A, Vol. I (1986) 303-414.

[29] E. D'Hoker and D.H. Phong, "Order parameters, free fermions, and conservation laws for Calogero-Moser systems", hep-th/9808156, to appear in Asian J. Math.

[30] E. D'Hoker, I.M. Krichever, and D.H. Phong, "The renormalization group equation for $\mathcal{N}=2$ supersymmetric gauge theories", Nucl. Phys. **B 494** (1997) 89-104, hep-th/9610156.

[31] E. D'Hoker, I.M. Krichever, and D.H. Phong, "The effective prepotential for $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ gauge theories", Nucl. Phys. **B489** (1997) 179, hep-th/9609041;
E. D'Hoker, I.M. Krichever, and D.H. Phong, "The effective prepotential for $\mathcal{N} = 2$ supersymmetric $SO(N_c)$ and $Sp(N_c)$ gauge theories", Nucl. Phys. **B 489** (1997) 211-222, hep-th/9609145
E. D'Hoker and D.H. Phong, "Strong Coupling Expansions of SU(N) Seiberg-Witten Theory", Phys. Lett. **B397** (1997) 94; hep-th/9701055.

[32] J. Minahan, D. Nemeschansky, and N. Warner, "Instanton expansions for mass deformed $\mathcal{N} = 4$ super Yang-Mills theory", hep-th/9710146.

[33] M. Marino and G. Moore, "The Donaldson-Witten Function for Gauge Groups of rank larger than one", hep-th/9802185;
M. Marino, "The Uses of Whitham Hierarchies", hep-th/9905053.

[34] K. Vaninsky, "On explicit parametrization of spectral curves for Moser-Calogero particles and its applications", December 1998 preprint.

[35] E. Martinec and N. Warner, "Integrable systems and supersymmetric gauge theories", Nucl. Phys. **B 459** (1996) 97-112, hep-th/9509161.

[36] A. Gorsky, "Branes and Integrability in the $\mathcal{N} = 2$ SUSY YM theory", Int. J. Mod. Phys. **A12** (1997) 1243, hep-th/9612238;
A. Gorsky, S. Gukov, A. Mironov, "SUSY field theories, integrable systems and their stringy brane origin", hep-th/9710239;
A. Cherkis and A. Kapustin, "Singular monopoles and supersymmetric gauge theories in three dimensions", hep-th/9711145.